

# The Rational Distance Problem for Equilateral Triangles

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## §-1 Abstract

Let  $(P)$  denote the problem of existence of a point in the plane of a given triangle  $T$ , that is at rational distance from all the vertices of  $T$ . Answer to  $(P)$  is positive if  $T$  has a rational side and the square of all sides are rational (see [1]). In [2], a complete solution to  $(P)$  is given for all isosceles triangles with one rational side. In this article, we provide a complete solution to  $(P)$  for all equilateral triangles.

In all what follows,  $\theta$  denotes an arbitrary positive real number and  $T = [\theta]$  denotes the equilateral triangle with side-length  $\theta$ . For convenience, we say that  $\theta$  is "good" (or "suitable") if answer to  $(P)$  is positive for the triangle  $T = [\theta]$ . Clearly, the property " $\theta$  is good" is invariant by any rational re-scaling of  $\theta$ .

It turns out that the *good*  $\theta$  must have algebraic degree 1, 2, or 4, and they form a subclass of the *positive* bi-quadratic numbers, that is, the positive roots of equations of the form  $x^4 + ux^2 + v = 0$ ,  $u, v \in \mathbb{Q}$ . The general form of such numbers is

$$\sqrt{\alpha \pm \sqrt{\beta}}, \quad \alpha, \beta \in \mathbb{Q}, \quad \beta \geq 0, \quad \alpha \pm \sqrt{\beta} \geq 0$$

that includes positive numbers of the form

$$\alpha, \sqrt{\alpha}, \alpha \pm \sqrt{\beta}, \sqrt{\alpha} \pm \sqrt{\beta}, \quad \alpha, \beta \in \mathbb{Q}, \quad \alpha, \beta \geq 0.$$

**Notations and conventions:**  $(x, y)$  and  $(x, y, z)$  denote the g.c.d.  $\left(\frac{x}{p}\right)$  denotes Legendre's symbol. A triangle with side-lengths  $a, b, c$  is denoted by  $T = [a, b, c]$ . A triangle is non-degenerated if it has positive area. A radical is non-degenerated if it is irrational.

## §-2 The results

**Theorem 0** *If  $\theta$  is good, then,  $\theta$  is bi-quadratic. More precisely,  $\theta^2 = \alpha \pm \sqrt{\beta}$  for some  $\alpha, \beta \in \mathbb{Q}$ ,  $\beta \geq 0$ , and  $\alpha$  positive.*

**Theorem 1** *Suppose  $\theta \notin \mathbb{Q}$  and  $\theta^2 \in \mathbb{Q}$ . Then,  $\theta$  is good  $\Leftrightarrow \theta$  has the form  $\theta = \lambda \sqrt{p_1 \dots p_r}$  where  $\lambda \in \mathbb{Q}$ ,  $\lambda > 0$ ,  $r \geq 1$ ,  $p_1, \dots, p_r$  are distinct odd primes,  $p_i$  is either 3 or of the form  $6k + 1$ .*

**Theorem 2** *Suppose  $\theta^2 = \alpha \pm \sqrt{\beta}$ ,  $\alpha, \beta \in \mathbb{Q}$ ,  $\alpha, \beta > 0$ ,  $\sqrt{\beta} \notin \mathbb{Q}$ . Then,  $\theta$  is good  $\Leftrightarrow$  up to a rational re-scaling of  $\theta$ ,  $\theta$  is described as follows:*

$$2\theta^2 = (a^2 + b^2 + c^2) \pm 4\Delta\sqrt{3}$$

where  $[a, b, c]$  is a non-degenerated primitive integral triangle with area  $\Delta$  such that  $4\Delta\sqrt{3} \notin \mathbb{Q}$ .

**Remark**  $\Delta$  is given by Hero's formula,  $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ ,  $s = \frac{1}{2}(a+b+c)$ . Equivalently,  $4\Delta\sqrt{3} = \sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}$ , and the condition  $4\Delta\sqrt{3} \notin \mathbb{Q}$  means that this latter radical is non-degenerated.

### §-3 Proofs of theorems 0 and 1

*Proof of theorem 0:* Suppose  $\theta$  good. Let  $M$  be a point in the plane of triangle  $T = [\theta]$ , whose distances from the vertices of  $T$  are all rational. the following fundamental relation is well-known (see[3]):

$$3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2 \quad (.)$$

Expanding  $(.)$  yields a relation as  $\theta^4 - u\theta^2 + v = 0$ , where  $u, v \in \mathbb{Q}$  and  $u = a^2 + b^2 + c^2 > 0$ . Solving for  $\theta^2$  yields  $\theta^2 = \alpha \pm \sqrt{\beta}$ , with  $\alpha, \beta \in \mathbb{Q}$  and  $\alpha = \frac{1}{2}u > 0$ .

**Lemma 1:** let  $q > 1$  be a square-free integer. Then we have:

The equation  $x^2 + 3y^2 = qz^2$  has a solution in integers  $x, y, z$ , with  $z \neq 0$  if and only if any prime factor of  $q$  is either 3 or of the form  $6k + 1$

**Proof:** Suppose first that  $q$  has only prime factors as 3 or  $6k + 1$ . Since the quadratic form  $x^2 + 3y^2$ ,  $x, y \in \mathbb{Z}$ , represents 3 and every prime  $p = 6k + 1$ , and since the set  $\{x^2 + 3y^2, x, y \in \mathbb{Z}\}$  is closed by multiplication, we conclude that the equation  $x^2 + 3y^2 = q.z^2$  has a solution in integers  $x, y, z$  with  $z = 1$ .

Conversely, suppose that  $x^2 + 3y^2 = q.z^2$  has a solution in integers  $x, y, z$ ,  $z \neq 0$ . Pick such a solution with  $|z|$  minimum. Clearly,  $(x, y) = 1$ . I claim that  $q$  is odd and has no prime factor  $6k - 1$ . For the purpose of contradiction, we consider two cases:

**case 1:**  $q$  is even. Set  $q = 2w, w$  odd. From  $x^2 + 3y^2 = 2wz^2$ , we see that  $x \equiv y \pmod{2}$ . As  $(x, y) = 1$ ,  $x$  and  $y$  must be odd, so  $x^2 + 3y^2 \equiv 4 \pmod{8}$ . Now,  $4/2wz^2$  yields  $wz^2$  even. But  $w$  is odd, hence  $z$  is even, so  $2wz^2 \equiv 0 \pmod{8}$ . We get a contradiction.

**case 2:**  $q = p.w$  for some prime  $p = 6k - 1$ .  $x^2 + 3y^2 = pwz^2$  yields  $x^2 + 3y^2 \equiv 0 \pmod{p}$ . As  $(x, y) = 1$ ,  $p$  cannot divide  $y$ . Hence for some  $t \in \mathbb{Z}$ ,  $yt \equiv 1 \pmod{p}$ . Therefore,  $x^2t^2 + 3y^2t^2 \equiv x^2t^2 + 3 \equiv 0 \pmod{p}$ , so  $-3 \equiv (xt)^2 \pmod{p}$ . Hence  $\left(\frac{-3}{p}\right) = +1$  contradicting  $p = 6k - 1$ .

**Lemma 2:** Let  $\theta = \lambda\sqrt{q}$ ,  $\lambda \in \mathbb{Q}$ ,  $\lambda > 0$ ,  $q > 1$  square-free integer. We have:  
 $\theta$  is good  $\Leftrightarrow$  There are  $a, b, e, r, s \in \mathbb{Q}$ ,  $e \neq 0$ , such that

$$a^2 + 3b^2 = q \quad (1)$$

$$(a + e)^2 + 3(b + e)^2 = qr^2 \quad (2)$$

$$(a - e)^2 + 3(b + e)^2 = qs^2 \quad (3)$$

**Proof:** By re-scaling, we take  $\theta = 2\sqrt{q}$ . Let  $T = ABC = [\theta]$ . Choose a  $x - y$  axis to get the coordinates  $A(0, \sqrt{3q})$ ,  $B(-\sqrt{q}, 0)$ ,  $C(\sqrt{q}, 0)$ .

• Suppose first that  $\theta$  is good: There is a point  $M = M(x, y)$  in the plane of  $T$  such that  $MA, MB, MC \in \mathbb{Q}$ . Clearly,  $M \neq A, B, C$ . Set  $w = \frac{MA}{q}$ ,  $r = \frac{MB}{wq}$ ,  $s = \frac{MC}{wq}$ . Then,  $w, r, s \in \mathbb{Q} - \{0\}$ . The Pythagoras relations are:

$$\overline{MA}^2 = x^2 + (y - \sqrt{3q})^2 = w^2q^2 \quad (1')$$

$$\overline{MB}^2 = (x + \sqrt{q})^2 + y^2 = w^2q^2r^2 \quad (2')$$

$$\overline{MC}^2 = (x - \sqrt{q})^2 + y^2 = w^2q^2s^2 \quad (3')$$

Subtracting (2') and (3') yields  $x = \frac{1}{4}w^2q(r^2 - s^2).\sqrt{q}$ , that is,

$$x = \alpha\sqrt{q}, \quad \alpha \in \mathbb{Q} \quad (4)$$

Then (2') gives  $y^2 \in \mathbb{Q}$ , and then (1') gives  $2y\sqrt{3q} \in \mathbb{Q}$ , hence,  $y = \gamma\sqrt{3q}$ ,  $\gamma \in \mathbb{Q}$ . For convenience, we put  $\gamma = \beta + 1$ , obtaining

$$y = (\beta + 1)\sqrt{3q}, \quad \beta \in \mathbb{Q} \quad (5)$$

Due to (4) and (5), equations (1'), (2'), (3') become after dividing by  $q$  :

$$\alpha^2 + 3\beta^2 = qw^2$$

$$(\alpha + 1)^2 + 3(\beta + 1)^2 = qw^2r^2$$

$$(\alpha - 1)^2 + 3(\beta + 1)^2 = qw^2s^2$$

Set  $a = \frac{\alpha}{w}$ ,  $b = \frac{\beta}{w}$ ,  $e = \frac{1}{w}$ . Dividing by  $w^2$ , we get precisely relations (1), (2), (3).

• Conversely suppose that relations (1), (2), (3) hold with some  $a, b, e, r, s \in \mathbb{Q}$ ,  $e \neq 0$ . Define point  $M = M(x, y)$  in the plane of  $T$  by

$$x = \frac{a}{e}\sqrt{q}, \quad y = \left(\frac{b}{e} + 1\right)\sqrt{3q}$$

We may write:

$$\overline{MA}^2 = x^2 + (y - \sqrt{3q})^2 = q\frac{a^2}{e^2} + 3q\frac{b^2}{e^2} = \frac{q}{e^2}(a^2 + 3b^2) = \frac{q}{e^2} \cdot q = \left(\frac{q}{e}\right)^2$$

$$\overline{MB}^2 = \left(\left(\frac{a+e}{e}\right)\sqrt{q}\right)^2 + \left(\left(\frac{b+e}{e}\right)\sqrt{3q}\right)^2 = \frac{q}{e^2}((a+e)^2 + 3(b+e)^2) = \frac{q}{e^2} \cdot qr^2 = \left(\frac{qr}{e}\right)^2$$

$$\overline{MC}^2 = \left(\left(\frac{a-e}{e}\right)\sqrt{q}\right)^2 + \left(\left(\frac{b+e}{e}\right)\sqrt{3q}\right)^2 = \frac{q}{e^2}((a-e)^2 + 3(b+e)^2) = \frac{q}{e^2} \cdot qs^2 = \left(\frac{qs}{e}\right)^2$$

Therefore,  $MA$ ,  $MB$ ,  $MC$  are all rational.

### **Proof of theorem 1**

Let  $\theta$  such that  $\theta \notin \mathbb{Q}$  and  $\theta^2 \in \mathbb{Q}$ :  $\theta$  can be written as  $\theta = \lambda\sqrt{q}$ ,  $\lambda \in \mathbb{Q}$ ,  $\lambda > 0$ ,  $q > 1$  square-free integer.

- Suppose first that  $q$  is even or has a prime factor  $6k - 1$ . By lemma 1,  $a^2 + 3b^2 = q$ ,  $a, b \in \mathbb{Q}$ , is impossible. Hence, relation (1) in lemma 2 fails, so  $\theta$  is not good.
- Suppose now that  $q$  has only prime factors as 3 or  $6k + 1$ . We show that  $\theta$  is good using the characterization of lemma 2:

By lemma 1, for some  $a, b \in \mathbb{Q}$ , we have  $a^2 + 3b^2 = q$ . Set  $e = -\frac{q}{4b} = \frac{-(a^2 + 3b^2)}{4b}$ ,  $r = \frac{a-b}{2b}$ ,  $s = \frac{a+b}{2b}$ . We have

$$\begin{aligned} (a+e)^2 + 3(b+e)^2 &= (a^2 + 3b^2) + 4e^2 + 2e(a+3b) = q + \frac{q^2}{4b^2} - \frac{q}{2b}(a+3b) \\ &= \frac{q}{4b^2}(4b^2 + q - 2b(a+3b)) = \frac{q}{4b^2}(4b^2 + a^2 + 3b^2 - 2ab - 6b^2) \\ &= \frac{q}{4b^2}(a^2 + b^2 - 2ab) = q\frac{(a-b)^2}{4b^2} = q.r^2 \quad \text{and} \end{aligned}$$

$$\begin{aligned} (a-b)^2 + 3(b+e)^2 &= (a^2 + 3b^2) + 4e^2 - 2e(a-3b) = q + \frac{q^2}{4b^2} + \frac{q}{2b}(a-3b) \\ &= \frac{q}{4b^2}(4b^2 + q + 2b(a-3b)) = \frac{q}{4b^2}(4b^2 + a^2 + 3b^2 + 2ab - 6b^2) \\ &= \frac{q}{4b^2}(a^2 + b^2 + 2ab) = q\frac{(a+b)^2}{4b^2} = q.s^2 \end{aligned}$$

## §-4 Proof of theorem 2

**Lemma 3:** Let  $x, y, z, t$  be positive real numbers such that

$$3(x^4 + y^4 + z^4 + t^4) = (x^2 + y^2 + z^2 + t^2)^2 \quad (\odot)$$

Then, any three of  $x, y, z, t$  satisfy the triangle inequality.

**Proof:** Since  $x, y, z, t$  play symmetric roles, it suffices to show that  $x, y, z$  satisfy the triangle inequality. Write  $(\odot)$  as

$$t^4 - (x^2 + y^2 + z^2)t^2 + (x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2) = 0$$

The discriminant  $\Delta$  of this trinomial in  $t^2$  must be non-negative. But,  $\Delta = 6(x^2y^2 + y^2z^2 + z^2x^2) - 3(x^4 + y^4 + z^4)$  that factors as  $\Delta = 3(x + y + z)(-x + y + z)(x - y + z)(x + y - z)$ . Hence,  $(-x + y + z)(x - y + z)(x + y - z) \geq 0$ . The reader can easily check (using contraposition) that  $x, y, z$  must satisfy the triangle inequality.

**Lemma 4:** Let  $T = ABC = [\theta]$ . Let  $a, b, c$  be positive real numbers satisfying

$$3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2$$

Then, there is a point  $M$  in the plane of  $T$  such that  $MA = a$ ,  $MB = b$ , and  $MC = c$ .

**Proof:** By lemma 3,  $a, b$ , and  $\theta$  satisfy the triangle inequality. In particular,  $a + b \geq \theta$ . It follows that the circle  $\mathcal{C}(A, a)$  intersects the circle  $\mathcal{C}(B, b)$  at two points  $M_1$  and  $M_2$  ( $M_1 = M_2$  if  $a + b = \theta$ ). Set  $c_1 = M_1C$  and  $c_2 = M_2C$ . By the fundamental relation  $(\bullet)$  we have  $3(a^4 + b^4 + c_1^4 + \theta^4) = (a^2 + b^2 + c_1^2 + \theta^2)^2$  and  $3(a^4 + b^4 + c_2^4 + \theta^4) = (a^2 + b^2 + c_2^2 + \theta^2)^2$ . Therefore,  $c_1^2$  and  $c_2^2$  are the roots of the trinomial in  $T$

$$T^2 - (a^2 + b^2 + \theta^2)T + (a^4 + b^4 + \theta^4 - a^2b^2 - b^2\theta^2 - \theta^2a^2) = 0$$

Since by hypothesis  $c^2$  is also a root of this trinomial, we must have  $c^2 = c_1^2$  or  $c^2 = c_2^2$ . Hence  $c = c_1$  or  $c = c_2$ . Therefore,  $a, b$  and  $c$  are the distances from either point  $M_1$  or  $M_2$  to the vertices  $A, B$  and  $C$  of  $T$ .

**Proof of theorem 2:**

Let  $\theta > 0$  such that  $\theta^2 = \alpha \pm \sqrt{\beta}$ ,  $\alpha, \beta \in \mathbb{Q}$ ,  $\alpha, \beta > 0$ ,  $\sqrt{\beta} \notin \mathbb{Q}$ .

• Suppose first that  $\theta$  is good: let  $P$  be a point in the plane of  $T = ABC = [\theta]$  such that  $PA = a$ ,  $PB = b$ ,  $PC = c$  are all rational. We have

$$3(a^4 + b^4 + c^4 + \theta^4) = (a^2 + b^2 + c^2 + \theta^2)^2 \quad (\bullet)$$

By lemma 3,  $a, b$ , and  $c$  satisfy the triangle inequality. Relation  $(\bullet)$  yields

$$\begin{aligned} \theta^4 - U\theta^2 + V &= 0 \quad \text{with } U = a^2 + b^2 + c^2 \quad \text{and} \\ V &= a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \quad (U, V \in \mathbb{Q}). \end{aligned}$$

Solving for  $\theta^2$ , we get

$$2\theta^2 = (a^2 + b^2 + c^2) \pm \sqrt{3(a + b + c)(-a + b + c)(a - b + c)(a + b - c)} \quad (\star)$$

Since  $\theta^2$  has algebraic degree 2, then, the radical in  $(\star)$  is non-degenerated. In particular, the triangle  $[a, b, c]$  is non degenerated. Select a sufficiently large positive integer  $N$  such that  $Na, Nb, Nc$  are all integers and set  $D = (Na, Nb, Nc)$ . If we multiply relation  $(\star)$  by  $\frac{N^2}{D^2}$ , this results in replacing in  $(\star)$   $\theta$  by  $\frac{N}{D}.\theta$  and  $a, b, c$  by the integers  $\frac{Na}{D}, \frac{Nb}{D}, \frac{Nc}{D}$  respectively. As an outcome, we obtain *essentially* the same relation  $(\star)$  where  $\theta$  has been re-scaled by the rational  $\frac{N}{D}$ , and where the new symbols  $a, b, c$  represent relatively prime *positive* integers, satisfying the triangle inequality.

• Conversely, suppose that for some positive rational  $\lambda$ ,  $\theta_0 = \lambda.\theta$  is described precisely as in theorem 2. Eliminating the radical

$$4 \triangle \sqrt{3} = \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}$$

in the relation  $2\theta_0^2 = (a^2 + b^2 + c^2) \pm 4 \triangle \sqrt{3}$  leads to

$$3(a^4 + b^4 + c^4 + \theta_0^4) = (a^2 + b^2 + c^2 + \theta_0^2)^2$$

By lemma 4 there is a point  $M$  in the plane of  $T = [\theta_0]$  that is at distances  $a, b, c$  from the vertices of  $T$ . Since  $a, b, c$  are integers, then,  $\theta_0$  is good. Therefore,  $\theta = \lambda^{-1}\theta_0$  is also good.

We end this article with a few exercises:

1. Check which are "good" among the radicals:  $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}$ .
2. Show that the positive real number  $\theta = \sqrt{25 + 12\sqrt{3}}$  is "good".
3. Suppose that  $2\theta^2 = \alpha + \sqrt{\beta}$ ,  $\alpha, \beta \in \mathbb{Q}$ ,  $\alpha, \beta > 0$ ,  $\sqrt{\beta} \notin \mathbb{Q}$ , and  $\alpha^2 < \beta$ . Show that  $\theta$  is not good.
4. Produce solution-points to problem (P) for the triangle  $T = [\sqrt{3}]$ .
5. Let  $\theta = \alpha + \beta\sqrt[4]{q} > 0$ ,  $\alpha, \beta \in \mathbb{Q}$ ,  $\beta \neq 0$ ,  $q > 1$  square-free integer. Show that  $\theta$  is not good.
6. Suppose that  $2\theta^2 = \alpha \pm \sqrt{\beta} > 0$ ,  $\alpha, \beta \in \mathbb{Q}$ ,  $\alpha, \beta > 0$ ,  $\sqrt{\beta} \notin \mathbb{Q}$ . Write the fraction  $\alpha$  in *lowest terms* as  $\alpha = \frac{m}{n}$  ( $m, n$  positive integers) and suppose that  $mn$  has the form  $mn = 4^l(8k + 7)$ ,  $k, l$  non-negative integers. Then, prove that  $\theta$  is not good.

## References

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